Solutions

1. Determine if the vectors are linearly independent, and justify your answer.

i. $\begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\-8\end{bmatrix}, \begin{bmatrix}-1\\3\\1\end{bmatrix}$ Find solutions to vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$.

$$\left[\begin{array}{ccc|c}0&0&-1&0\\2&0&3&0\\3&-8&1&0\end{array}\right] \overset{r_1\leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c}2&0&3&0\\0&0&-1&0\\3&-8&1&0\end{array}\right] \overset{r_2\leftrightarrow r_3}{\sim} \left[\begin{array}{ccc|c}2&0&3&0\\3&-8&1&0\\0&0&-1&0\end{array}\right] \sim$$

$$\underset{-\frac{3}{2}r_1+r_2}{\overset{-r_3}{\sim}} \left[\begin{array}{ccc|c}2&0&3&0\\0&-8&-7/2&0\\0&0&1&0\end{array}\right] \underset{-\frac{1}{8}r_2}{\overset{\frac{1}{2}r_1}{\sim}} \left[\begin{array}{ccc|c}1&0&3/2&0\\0&1&7/16&0\\0&0&1&0\end{array}\right] \underset{-\frac{7}{16}r_3+r_2}{\overset{-\frac{3}{2}r_3+r_1}{\sim}} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$\Rightarrow x_1 = x_2 = x_3 = 0$ system has only trivial solution $\Rightarrow$ vectors are linear independent

ii. $\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}-3\\-9\end{bmatrix}$ Let $\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$, $\vec{v} = \begin{bmatrix}-3\\-9\end{bmatrix}$.

Then $u_1 = 3\cdot v_1$, but $u_2 = -3v_2$. So $\vec{u} \neq k\cdot\vec{v}$, and so the vectors are linearly independent.

2. Determine if the columns of the matrix form a linearly independent set, and justify your answer.

$$\begin{bmatrix}0&-3&9\\2&1&-7\\-1&4&-5\\1&-4&-2\end{bmatrix}$$ Solve matrix equation $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c}0&-3&9&0\\2&1&-7&0\\-1&4&-5&0\\1&-4&-2&0\end{array}\right] \overset{r_1\leftrightarrow r_4}{\sim} \left[\begin{array}{ccc|c}1&-4&-2&0\\2&1&-7&0\\-1&4&-5&0\\0&-3&9&0\end{array}\right] \underset{-2r_1+r_2}{\overset{r_1+r_3}{\sim}} \left[\begin{array}{ccc|c}1&-4&-2&0\\0&9&-3&0\\0&0&-7&0\\0&-3&9&0\end{array}\right] \underset{\frac{1}{7}r_3}{\overset{3r_2+r_4}{\sim}}$$

$$\sim \left[\begin{array}{ccc|c}1&-4&-2&0\\0&9&-3&0\\0&0&1&0\\0&0&0&0\end{array}\right] \overset{\frac{1}{9}r_2}{\sim} \left[\begin{array}{ccc|c}1&-4&-2&0\\0&1&-1/3&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$\Rightarrow$ Pivot position in every column, so there are no free variables

$\Rightarrow$ system has only the trivial solution $\Rightarrow$ column vectors of matrix A are linearly independent

3. For what values of h is $\vec{v}_3$ in the $Span\{\vec{v}_1, \vec{v}_2\}$?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

$\vec{v}_3 \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$ iff $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$ has a solution

$$\left[\begin{array}{cc|c} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{array}\right] \underset{-2r_1+r_3}{\overset{3r_1+r_2}{\sim}} \left[\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{array}\right]$$

$\Rightarrow$ gives equation $0=8$ so system is inconsistent independent of the value of h

$\Rightarrow \forall h \in \mathbb{R}$, $\vec{v}_3 \notin \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

4. Find the values of h for which the vectors are linearly dependent, and justify your answer.

i. $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$

Find h so that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has non-trivial solution.

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & h & 0 \end{array}\right] \underset{-2r_1+r_3}{\overset{r_1+r_2}{\sim}} \left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 4+h & 0 \end{array}\right] \underset{-\frac{1}{2}r_2}{\overset{\frac{1}{2}r_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 4+h & 0 \end{array}\right] \sim$$

$$\overset{r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4+h & 0 \end{array}\right]$$

- if $4+h \neq 0$, then pivot in each column and system has only trivial solution $x_1 = x_2 = x_3 = 0$.
- if $4+h=0$, then x_3 is free variable and system has non-trivial solution $\Rightarrow$ vectors are linearly dependent iff $\underline{h=-4}$.

ii. $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & 3 & 0 \end{array}\right] \underset{\frac{1}{3}r_1}{\overset{2r_1+r_2}{\sim}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -8 & h+18 & 0 \\ 1 & -3 & 3 & 0 \end{array}\right] \overset{-r_1+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -8 & h+18 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \sim$$

$$\underset{-r_2}{\overset{r_2 \leftrightarrow r_3}{\sim}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -8 & h+18 & 0 \end{array}\right] \overset{8r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h+18 & 0 \end{array}\right]$$

- if $h \neq -18$, then only trivial sol.
- if $h=-18$, x_3 is a free variable

So system has non-trivial solution $\Rightarrow$ vectors are linearly dependent iff $h=-18$.

5. Determine by inspection whether the vectors are linearly independent, and justify your answer.

i. $\begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}2\\8\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-1\\7\end{bmatrix}$ $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent (2 equations with 4 variables)

ii. $\begin{bmatrix}2\\-4\\8\end{bmatrix}, \begin{bmatrix}-3\\6\\-12\end{bmatrix}$ $\vec{v}_2 = -\frac{3}{2}\vec{v}_1$ $\vec{v}_1, \vec{v}_2$ are linearly dependent

iii. $\begin{bmatrix}5\\-3\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\-2\\4\end{bmatrix}$ $\vec{v}_2 = \vec{0}$, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent

$0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0}$

iv. $\begin{bmatrix}-8\\12\\-4\end{bmatrix}, \begin{bmatrix}2\\-3\\-1\end{bmatrix}$ $\vec{v}_2 \neq k \cdot \vec{v}_1$, so $\vec{v}_1, \vec{v}_2$ are linearly independent

v. $\begin{bmatrix}1\\4\\-7\end{bmatrix}, \begin{bmatrix}-2\\5\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ $\vec{v}_3 = \vec{0}$, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent

$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \vec{0}$

vi. $\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}-1\\5\end{bmatrix}$ $\vec{v}_1 \neq k \cdot \vec{v}_2$ so $\vec{v}_1, \vec{v}_2$ are linearly independent

6. Construct a possible echelon form of the matrix with:

i. A is a 2×2 matrix with linearly dependent columns.

$\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} \rightarrow \vec{a}_2 = 2\,\vec{a}_1$

ii. A is a 3×3 matrix with linearly independent columns.

$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ → pivot position in every column

⇒ $[A \mid \vec{0}]$ gives only trivial solution to the equation $A\vec{x} = \vec{0}$.

iii. A is a 3×2 matrix such that $A\bar{x}=\bar{0}$ has a non-trivial solution.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\vec{a}_2 = 0$

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$ $\vec{a}_2 = 3\vec{a}_1$

$x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 = \vec{0}$ has non-trivial solution

$\Leftrightarrow$ $\vec{a}_1, \vec{a}_2$ are linearly dependent

iv. A is a 3×2 matrix such that $A\bar{x}=\bar{0}$ has only the trivial solution.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{a}_1 \neq k \cdot \vec{a}_2$

$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 9 \end{bmatrix}$ $\vec{a}_1 \neq k \cdot \vec{a}_2$

$x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 = \vec{0}$ has only trivial solution

$\Leftrightarrow$ $\vec{a}_1, \vec{a}_2$ are linearly independent

$\Leftrightarrow$ $\nexists\, k \in \mathbb{R}$ s.t. $\vec{a}_2 = k \cdot \vec{a}_1$

7. Given $A = \begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}$, observe that the first column minus 3 times the second column equals the third column. Find a non-trivial solution to the equation $A\bar{x}=\bar{0}$.

$\vec{a}_1 - 3\vec{a}_2 = \vec{a}_3 \Rightarrow \vec{a}_1 - 3\vec{a}_2 - \vec{a}_3 = \vec{0} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -3 \\ x_3 = -1 \end{cases}$

$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}$ has solution (x_1, x_2, x_3).

$\Updownarrow$

$A\vec{x} = \vec{0}$

8. Mark the following statements as true or false. If the statement is true, provide a short proof for it. If the statement is false, provide a ***counterexample*** for it.

i. If $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ are in R^4 and $\bar{v}_3 = 2\bar{v}_1 + \bar{v}_2$, then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is linearly dependent. TRUE

Note $2\vec{v}_1 + 1\cdot\vec{v}_2 - 1\cdot\vec{v}_3 + 0\cdot\vec{v}_4 = \vec{0}$,

So $x_1\cdot\vec{v}_1 + x_2\cdot\vec{v}_2 + x_3\cdot\vec{v}_3 + x_4\cdot\vec{v}_4 = \vec{0}$ has a non-trivial solution $(2, 1, -1, 0)$.

Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

ii. If $\vec{v}_1$ and $\vec{v}_2$ are in R^4, and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. FALSE

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so } \vec{v}_2 \neq k \cdot \vec{v}_1$$

$0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0} \rightarrow \vec{v}_1, \vec{v}_2$ are linearly dependent

(Note that $\vec{v}_1 = 0 \cdot \vec{v}_2$)

iii. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in R^3, and $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. FALSE

$$\vec{v}_3 \neq c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$-2\vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0} \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. dependent

iv. If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are in R^4 and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is also linearly dependent. TRUE

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. dependent $\Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ where not all $c_i = 0$. Let $c_4 = 0$.

Thus $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 =$

$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 0 \cdot \vec{v}_4 =$

$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Hence $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are lin. dependent

v. Suppose A is a $m \times n$ matrix with the property that for all $\vec{b}$ in R^m, the equation $A\vec{x} = \vec{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

Take $\vec{b} = \vec{0}$. Then by assumption $A\vec{x} = \vec{0}$ has at most one solution. However, since $A\vec{x} = \vec{0}$ always has the trivial solution $\vec{x} = \vec{0}$, we have that $A\vec{x} = \vec{0}$ has only the trivial solution.

Thus $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}$ has the only solution $x_1 = x_2 = \dots = x_n = 0$. Hence $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly independent.